UCLA

Birational Geometry Seminar

# Geography of surfaces with big cotangent bundle

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joint past work with Y. Asega, M.L. Weiss

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$X$ sm projective mfd of dimension $n$

$\Omega^1_X$ big $\Leftrightarrow h^0_\Omega(X) := \lim_{m\to\infty} \frac{h^0(X, S^m\Omega^1_X)}{m^{2n-1}} \neq 0$

$$\Omega^1_X \text{ big} \xRightarrow{\text{(Campana, Paun 15)}} X \text{ of general Type}$$

Let $A$ ample divisor on $X$

$\Omega^1_X$ big $\Rightarrow h^0(X, S^m\Omega^1_X \otimes \mathcal{O}(-A)) \nearrow m^{2n-1}$

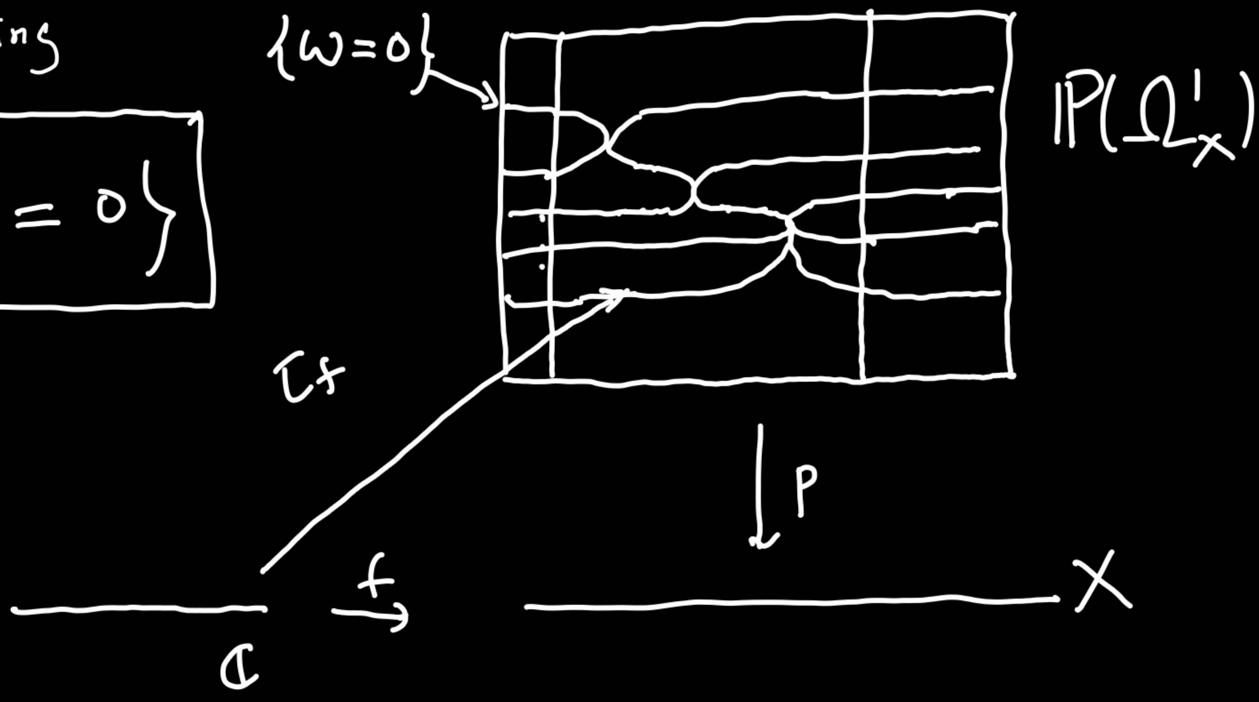
$$\underset{\omega}{\overset{\cup}{\ }} \quad \text{(sym. diff)}$$

entire curve, $f: \mathbb{C} \to X$ hol. nonconst.

$\omega$ gives an algebraic differential equation of the 1st order That $f$ must satisfy.

fund. vanishing

$$\boxed{\Gamma_f \subset \{\omega = 0\}}$$




<u>McQuillan (98)</u>: $X$ sm surf.

$$\Omega^1_X \text{ big}$$
$$\Downarrow$$
$$X \text{ satisfies:}$$

<u>Green-Griffiths-Lang</u> (GGL) conj:

$X$ proj var. of gen. type, Then $\exists\, Z \subsetneq X$ subvar s.t. $\forall$ entire curves $f : \mathbb{C} \to X$

$$f(\mathbb{C}) \subset Z.$$

- This approach is strengthned by using $k$-jet differentials with $k>1$ ($k=1 \Leftrightarrow$ sym. diff)

- Questions coming from the leading hyperbolicity conj. motivated this work, but the goal is to understand $\Omega_X^1$ big (surface case).

(Kobayashi conj 70) The generic hypersurface $X \subset \mathbb{P}^n$ $n \geq 3$, of degree $d \geq 2n-1$ has no entire curves.

- McQuillan, Demailly, El Goul, Paun etc proved Kobayashi conj. for very generic if $d \geq 18$.

- Siu, Brotbek proved for generic if $d \gg 0$.

- $P^s_m(X) := h^0(X, S^m\Omega^1_X)$ sym. m-genus.

birational inv.

$\Downarrow$

$\Omega^1_X$ big is a birational property

- $\Omega^1_X$ big is NOT preserved by deformations

(Bruckmann 71) $H \subset \mathbb{P}^n$, $n \geq 3$, smooth hypersurf.

$$P^s_m(H) = 0 \quad \forall\, m \geq 1$$

Q: Are there $X$ deformation equiv. to $H_d \subset \mathbb{P}^3$, hypersurf of degree $d$ with $\Omega^1_X$ big?

If yes, what is $d_{min} := \min\{d \mid \exists\, X \sim H_d, \Omega^1_X \text{ big}\}$?

Yes [Bogomolov, —, 06]

From now on we are in dim=2.

RRH + Bogomolov's vanishing ($h^2(X, S^m \Omega^1_X) = 0$, $m \geq 3$)

$$h^0_\Omega(X) = \frac{s_2(X)}{3!} + h^1_\Omega(X)$$

$s_2(X) = c_1^2(X) - c_2(X)$

$$h^1_\Omega(X) = \lim_{m \to \infty} \frac{h^1(X, S\Omega^1_X)}{m^3}$$

• $s_2(X) > 0 \Rightarrow \Omega^1_X$ big

$\Updownarrow$

$c_1^2/c_2 > 1$

$(s_2(H_d) = -4d^2 + 10d)$

Q: Can $\Omega^1_X$ be big if $c_1^2/c_2(X) \leq 1$?

Only interesting for minimal surf.

otherwise it has a trivial answer, since:

$S' \xrightarrow{\sigma} S$ blow up

$c_1^2(S') = c_1^2(S) - 1$

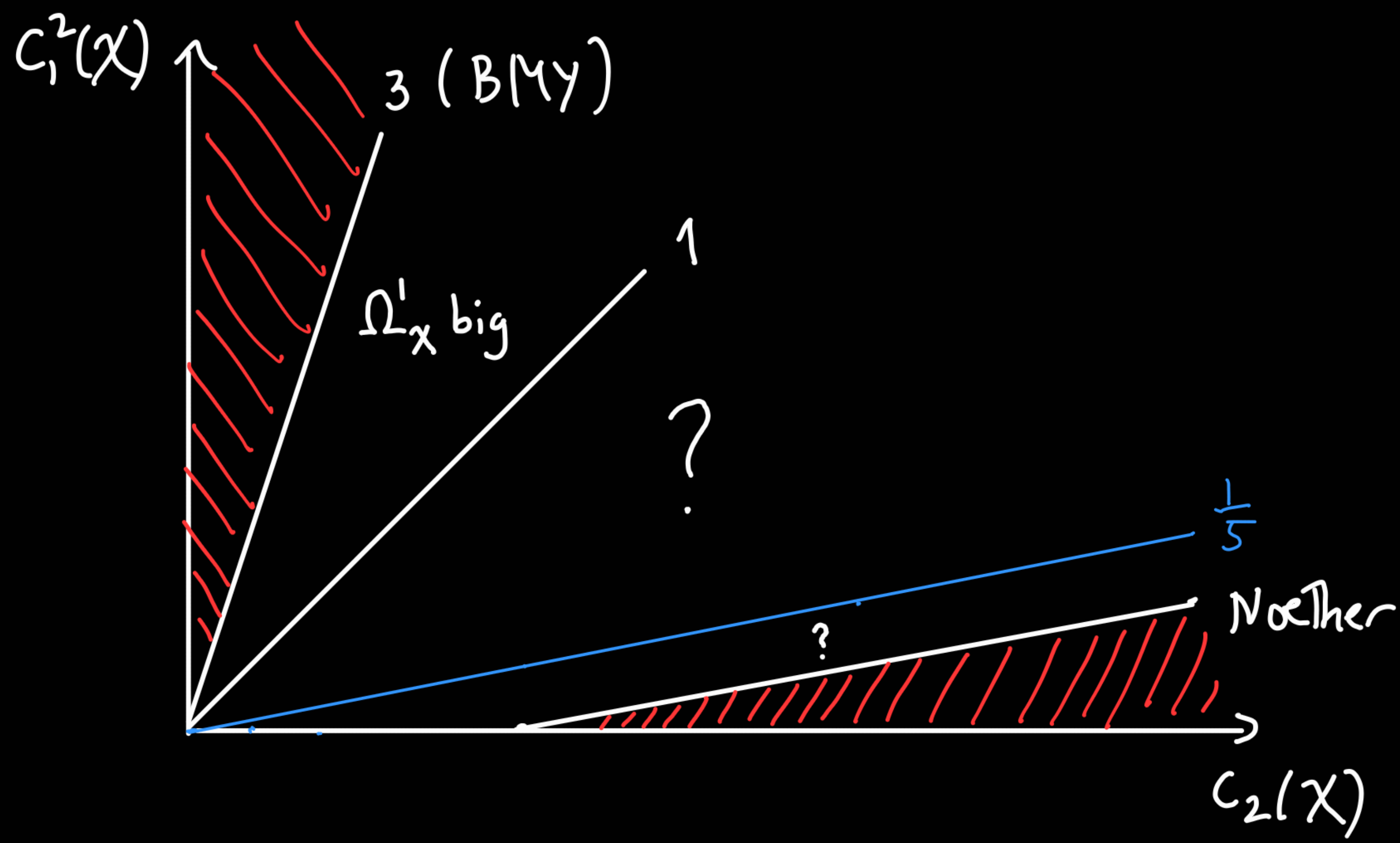
$c_2(S') = c_2(X) + 1$

$\mathcal{X}$ birational class of surf. of g.t.

$\mathcal{X}$ has 2 key models $\begin{cases} \nearrow X_{min} & (K_{X_{min}} \text{ nef, smooth}) \\ \searrow X_{can} & (K_{X_{can}} \text{ ample, ADE sing}) \end{cases}$

Set $c_1^2(\mathcal{X}) := c_1^2(X_{min})$ ; $c_2(\mathcal{X}) := c_2(X_{min})$

$$\Omega^1_{\mathcal{X}} \text{ big} := \Omega^1_X \text{ big}, \ X \text{ sm} \in \mathcal{X}$$



Understanding bigness of $\Omega^1_X$ for $c^2_1/c_2(X) \leq 1$ requires a hold of

$$h^1_\Omega(X) := h^1_\Omega(X_{min})$$

- (Asega, –, Weiss 19, 23) $X$ of gen. Type

$$h^1_\Omega(X_{min}) = NLh^1_\Omega(X) + Lh^1_\Omega(X)$$

$$NLh^1_\Omega(X) = \lim_{m\to\infty} \frac{h^1(X_{can}, \sigma_* S^m \Omega^1_{X_{min}})}{m^3}, \qquad Lh^1_\Omega(X) = \sum_{x \in Sing(X_{can})} h^1_\Omega(x)$$

$h^1_\Omega(x)$ surf. inv

$$h^1_\Omega(x) = \lim_{m\to\infty} \frac{h^1(\tilde{U}_x, S^m \Omega^1_{X_{min}})}{m^3}$$

$\tilde{U}_x \xrightarrow{\sigma} U_x$ min

$U_x$ germ of neigh of $x$

So we get the lowerbound:

$$Lh^1_\Omega(X) \leq h^1_\Omega(X)$$

<u>Canonical model sing. criterion:</u>

$X$ birational class of surf of gen. type

$$\underbrace{\sum_{x \in \mathrm{Sing}(X_{can})} h^1_\Omega(x) + \frac{s_2(X)}{3!}}_{\text{only topological and sing. data}} > 0 \Longrightarrow \Omega^1_X \text{ big}$$

## • How to find $h^1_\Omega(x)$?

Using Wahl, Blache, Langer work on Chern classes of orbifold vector bundles on orbifold surfaces and asymp. RR.

We can derive $h^1_\Omega(x)$ from

$$\hbar^0_\Omega(x) := \lim_{m\to\infty} \frac{\hbar^0_\Omega(x,m)}{m^3}$$

with

$$\hbar^0_\Omega(x,m) = \dim \left[ \frac{H^0(\tilde{U}_x \setminus E, S^m \Omega^1_{\tilde{U}_x})}{H^0(\tilde{U}_x, S^m \Omega^1_{\tilde{U}_x})} \right]$$

$\boxed{\text{obstructions to extend sym. diff. on } \tilde{U}_x \setminus E \text{ to } \tilde{U}_x.}$

$\tilde{U}_x \longrightarrow U_x$ min resol. of germ of the sing $x$.

Fundamental relation

$$\boxed{h^1_\Omega(x) = \frac{-s_{2,loc}(x)}{3!} - h^0_\Omega(x)}$$

<u>$x$ can. surf sing.</u>

$$\begin{cases} c^2_{1,loc}(x) = 0 \\ c_{2,loc}(x) = e(E_x) - \frac{1}{|G_x|} \end{cases}$$

$e(E_x)$ Top. Euler char. of $E_x$

$G_x$ loc. fund. gp at $x$

$\left(\begin{matrix} x \text{ canonical} \\ \text{sing.}\end{matrix}\right) \rightarrow$ $$\boxed{h^1_\Omega(x) = \frac{c_{2,loc}(x)}{3!} - h^0_\Omega(x)}$$

• <u>$(A,-,W)$</u> $x$ can. surf sing.

$$h^1_\Omega(x) \geq \frac{c_{2,loc}(x)}{2 \cdot 3!}$$

$$\left(\begin{matrix} \text{local cohom} \\ \Downarrow \\ h^0_\Omega(x,m) \leq h^1_\Omega(x,m)\end{matrix}\right)$$

• $[A, -, W]$ ($A_n$ singularities)

$h^0_\Omega(A_n, m)$ is given by the (polynomial) weighted lattice sum over a polygon $P_n(m)$:

\# of obstruction to extend sym. diffs of type $(i,k,m)$

$$h^0_\Omega(A_n, m) = \sum_{\substack{(i,k) \in P_n(m) \cap \mathbb{Z}^2 \\ i+(n+1)k \equiv_{(2)} m}} h^0_\Omega(A_n, i, k, m)$$

$P_n(m)$ is a polygon that for fixed $n$ varies with $m$ preserving slopes and up to $1^{st}$ order changes with $m$ as dilations.

We can use Ehrhart Theory and obtain:

$$\hbar^{\circ}_{\Omega}(A_n, m) \text{ is a quasi-polynomial of degree 3 in } m \text{ of the form:}$$

$$\| $$

$$\hbar^{\circ}_{\Omega}(A_n)\, m^3 + 3\, \hbar^{\circ}_{\Omega}(A_n)\, m^2 + c_n(m)\, m + d_n(m)$$

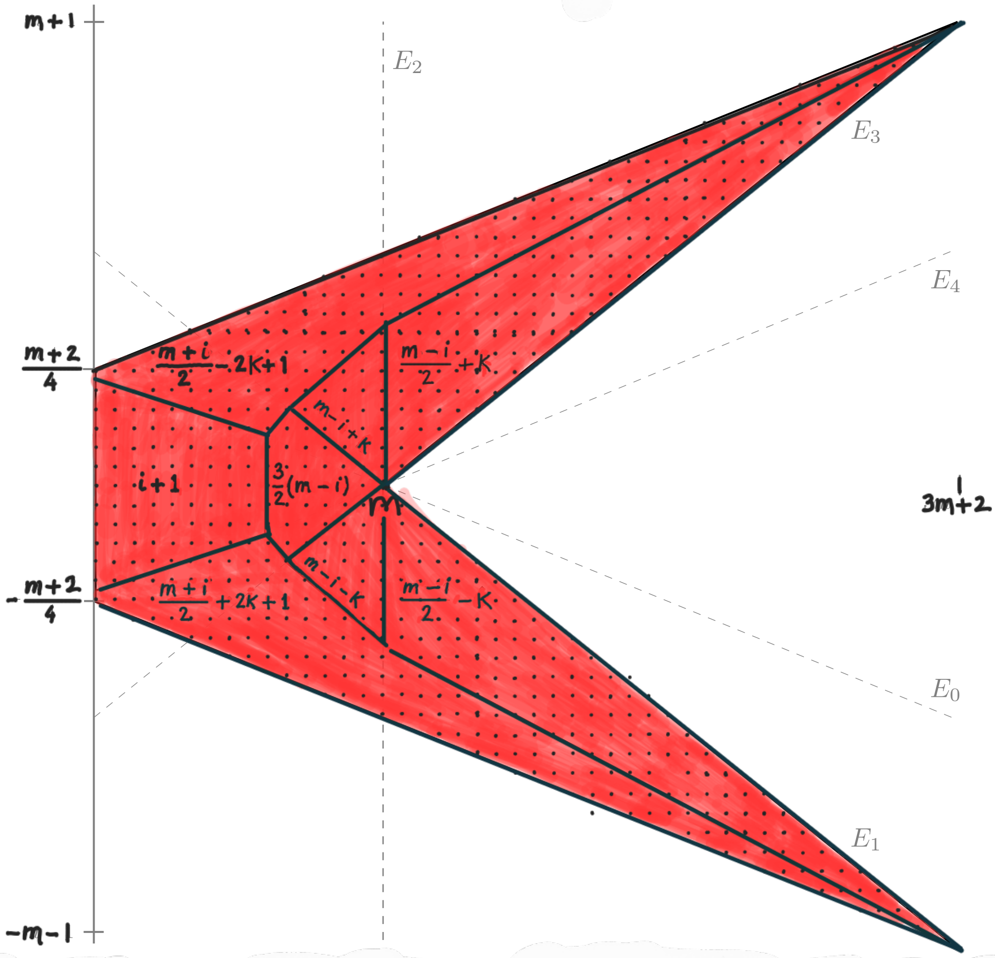
periodic in m.

• Ex: $(A_2)$

$$\hbar^{\circ}_{\Omega}(A_2, m) = \begin{cases} \frac{29}{216} m^3 + \frac{29}{72} m^2 + \frac{m}{12} & m \overset{6}{\equiv} 0 \\ \frac{29}{216} m^3 + \frac{29}{72} m^2 + \frac{m}{8} - \frac{143}{216} & m \overset{6}{\equiv} 1 \\ \frac{29}{216} m^3 + \frac{29}{72} m^2 + \frac{7m}{36} - \frac{2}{27} & m \overset{6}{\equiv} 2 \\ \frac{29}{216} m^3 + \frac{29}{72} m^2 + \frac{m}{8} + \frac{3}{8} & m \overset{6}{\equiv} 3 \\ \frac{29}{216} m^3 + \frac{29}{72} m^2 + \frac{m}{12} - \frac{10}{27} & m \overset{6}{\equiv} 4 \\ \frac{29}{216} m^3 + \frac{29}{72} m^2 + \frac{17m}{72} - \frac{7}{216} & m \overset{6}{\equiv} 5 \end{cases}$$

# $A_3$

$P_3(m)$ with polygonal subdivison where the weight function has a single expression.



- $\underline{(A, \rightarrow, W)}$

1) Closed formula for $h'_\Omega(A_n)$

$$h'_\Omega(A_n) = \frac{n^5+19n^4+83n^3+137n^2+80n}{6(n+1)^2(n+2)^2} - \frac{4}{3}\sum_{k=1}^{n}\frac{1}{k^2}$$

$$\Downarrow$$

i) $$\frac{n-\frac{1}{6}}{6} < h'_\Omega(A_n) < \frac{n}{6}$$

ii) $$h'_\Omega(A_{n+m}) > h'_\Omega(A_n) + h'_\Omega(A_m)$$

iii) $$\lim_{n\to\infty} \frac{\frac{c_{2,loc}(A_n)}{3!}}{h'_\Omega(A_n)} = 1$$

• Roulleau - Rousseau (14) gave a bigness criterion which can be reformulated in a form similar to the CMS-criterion:

RR-criterion:

$$\sum_{x \in \mathrm{Sing}(X_{can})} \frac{c_{2,loc}(x)}{2 \cdot 3!} + \frac{s_2(X)}{3!} > 0 \Rightarrow \Omega^1_X \text{ big.}$$

Recall: CMS-criterion

$$\sum_{x \in \mathrm{Sing}(X_{can})} h^1_{\Omega}(x) + \frac{s_2(X)}{3!} > 0 \Rightarrow \Omega^1_X \text{ big}$$

larger! (can be up to 2x larger for the $A_n$ case)

We will see the geographic impact of this!

# Bounds on $Lh^1_\Omega(X)$

Let $X \in GTA = \left\{ \begin{array}{l} \text{bir. classes of surf of} \\ \text{gen. Type s.t } X_{can} \\ \text{only has sing. of Type } A \end{array} \right\}$.

$$\mathrm{Sing}(X_{can}) = \{A_{n_1}, \dots, A_{n_k}\}$$

Set $\rho_{-2}(X) := \#\{(-2) \text{ curves on } X_{min}\}$

$$\boxed{\sum_{i=1}^{k} n_i = \rho_{-2}(X)}$$

We have:

i) $h^1_\Omega(A_{n_1}) + \dots + h^1_\Omega(A_{n_k}) < h^1_\Omega(A_{n_1 + \dots + n_k})$

ii) $\dfrac{\sum_{i=1}^{k} n_i - k\left(\frac{1}{6}\right)}{6} < h^1_\Omega(A_{n_1}) + \dots + h^1_\Omega(A_{n_k})$

$$\Downarrow$$

$$\boxed{\frac{\rho_{-2}(X) - \frac{1}{6}\#\mathrm{Sing}(X_{can})}{6} < Lh^1_\Omega(X) < \frac{\rho_{-2}(X)}{6}}$$

$$\Downarrow$$

$$\rho_{-2}(X) - \frac{\# \operatorname{Sing}(X_{can})}{6} > -s_2(X) \overset{\text{CMS}}{\Longrightarrow} \Omega^1_X \text{ big}$$

$$\rho_{-2}(X) \leq -s_2(X) \Longrightarrow \text{CMS can't hold}$$

# Topological bounds on $\rho_{-2}(X)$

Miyaoka bound

$$\text{(MB)} \qquad \sum_{x \in \mathrm{Sing}(X_{can})} c_{2,loc}(x) \leq c_2(X) - \frac{1}{3} c_1^2(X)$$

addresses directly the sing. contribution of RR-criterion.

$\Rightarrow$ RR-criterion can only hold if $\frac{c_1^2}{c_2}(X) > \frac{3}{5}$

(e.g., would never hold for $X \sim H_d$, $d \leq 11$)

(MB) can be used to bound $\rho_{-2}(X)$, BUT it depends on the sing. involved.

it is quite strong if $(-2)$-curves come from $A_1$,

but not good if $(-2)$-curves come from a single singul.

- The $c_{2,loc}$ cost of a $(-2)$-curve on a $A_n, D_n, E_n$ sing. decreases with $n$.

- If $\operatorname{Sing}(X_{can}) = \{A_n\}$, then $(MB) \Rightarrow$

$$\rho_{-2}(X) = n < c_2(X) - \frac{1}{3}c_1^2(X) - \frac{1}{2}$$

- Only finitely many pairs $(c_2, c_1^2)$ would be out of reach of the CMS-criterion (the ones with $c_1^2 = 1, 2$) if this bound was sharp.

# Hodge Theory bound $\left(h^{1,1}(X_{min})\right)$

(HB) $\quad \rho_{-2}(X) \leqslant \rho(X_{min}) - 1 \leqslant \dfrac{5c_1^2(X) - c_2(X)}{6} + b_1(X) - 1$

$\uparrow$ Picard #

- If $b_1(X) = 0$, (HB) < (MB) if $c_1^2 \leq c_2$.

<u>(A, -, W)</u> If $X \in GTA$, Then

i) CMS-criterion can not hold if $c_1^2/c_2(X) \leq \frac{1}{5}$

ii) If $c_1^2 > \dfrac{c_2 + 7}{5}$, no known bound on $\rho_{-2}(X)$ can avoid CMS-criterion to hold.

- <u>Conjecture</u>:

$$\alpha_{big} := \inf \left\{ c_1^2/c_2(X) \;\middle|\; \Omega^1_X \text{ big} \right\} = \frac{1}{5}.$$

• To show $\alpha_{big} \simeq \frac{1}{5}$ we need contructions.

<u>regular case</u> (difficult, CMS-criterion key)

(De Oliveira, 24) The lowest $c_1^2/c_2$ known for regular surface $X$ with $\Omega_X^1$ big is $25/107$ (resolution of double cover of $\mathbb{P}^2$ branched along a curve with many ade sing)

dichotomy

<u>irregular case</u> (easier and does not need CMS-criterion)

(Sakai) $X$ surf of g.t mapping to curve $C$ with $g(C) \geq 2$ has big cotangent bundle.

(Debarre) Constructs examples of genus 2 fibrations over curves of genus $\geq 2$ with $c_1^2/c_2$ approaching $1/5$

<u>Cor</u>: $\alpha_{big} \leq 1/5$

– To show $\alpha_{big} \geq \frac{1}{5}$ (Work in progress with D. Brotbek and E. Rousseau)

## • Deformations of hypersurf, $H_d \subset \mathbb{P}^3$.

(A, –, W 23) $\forall d \geq 8$, $\exists X \sim H_d$ with $\Omega^1_X$ big.

For $d=7$, $\exists X \sim H_7$ with $\rho_{-2}(X) = 126$, CMS needs at least $\rho_{-2}(X) = 127$ (appearing in $\leq 6$ sing)

For $d=6$, CMS needs $\rho_{-2}(X) \geq 85$ and (HB) $\Rightarrow \rho_{-2}(X) \leq 85$.

For $d=5$, $c_1^2/c_2(H_5) = \frac{1}{11}$, CMS can not hold.

# Cyclic covers of $\mathbb{P}^2$ branched line arrangements in general position

$Y_{n,\nu}$ $n$-cyclic cover of $\mathbb{P}^2$ branched along $n\nu$ lines

$$\mathrm{Sing}(Y_{n,\nu}) = \frac{n\nu(n\nu-1)}{2} \quad A_{n-1} \text{ sing.}$$

$X_{n,\nu}$ min. resol. of $Y_{n,\nu}$

$$s_2(X_{n,\nu}) = \left(\frac{1}{n}-1\right)(n\nu)^2 - 3(n-1)n\nu + 6n$$

[A, –, W]: Cases CMS and RR criteria fail to hold $\Omega^1_{X_{n,\nu}}$ big

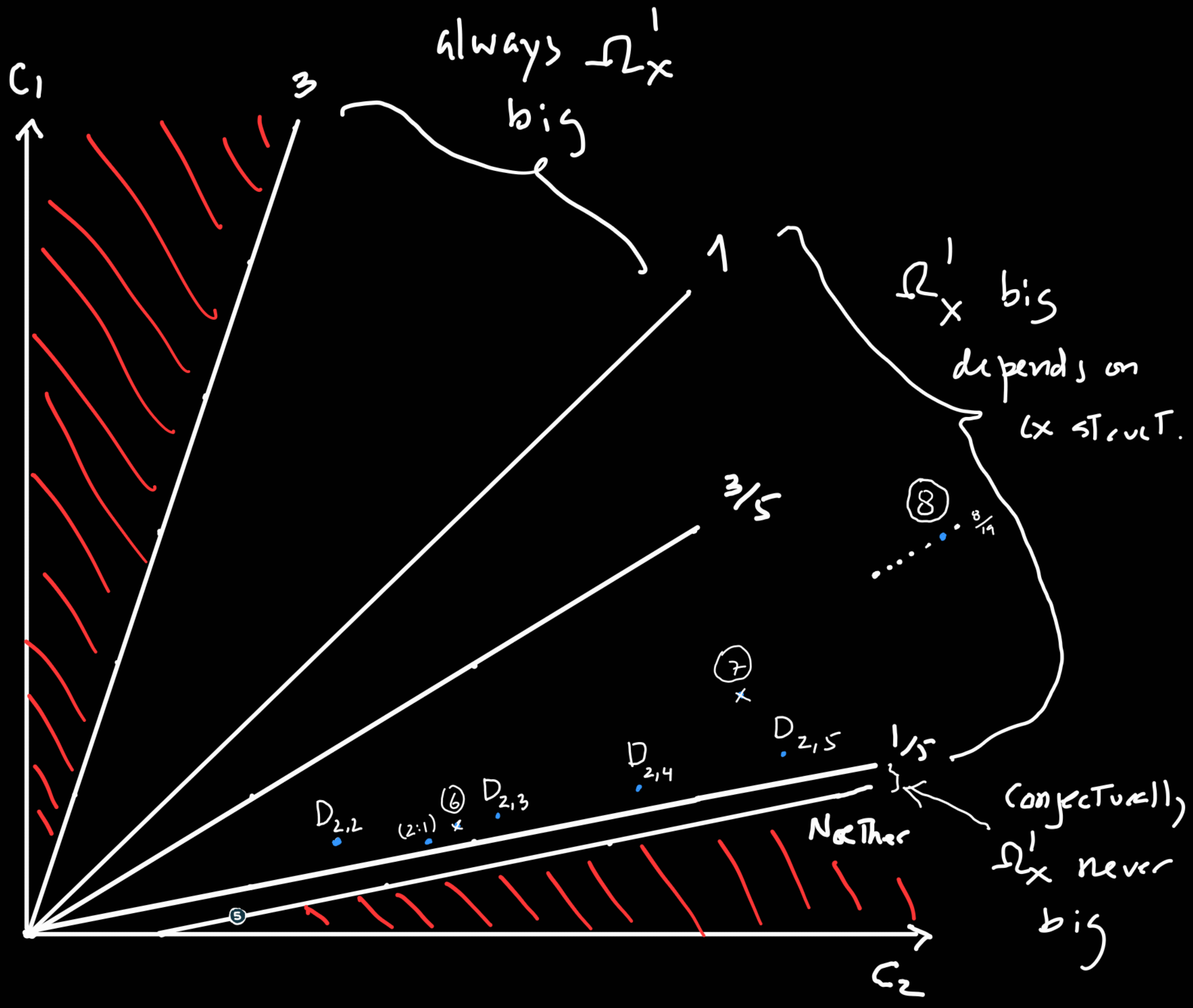
| $\nu \backslash n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $9 \leq n \leq 14$ |
|---|---|---|---|---|---|---|---|---|
| CMS | $\geq 1$ | $\leq 7$ | $\leq 3$ | 1 | 1 | 1 | | |
| RR | $\geq 1$ | $\geq 1$ | $\leq 12$ | $\leq 6$ | $\leq 3$ | $\leq 2$ | $\leq 2$ | 1 |

Corollary: $Y_{d,1}$ can be realized as an hypsurf in $\mathbb{P}^3$ of degree $d$.

above result gives examples, $X_{d,1} \sim H_d$

with $\Omega^1_{X_{d,1}}$ big if $d \geq 8$.

Bierskorn simult. resol.




The blue dots in the $(c_2, c_1^2)$-plane represent some examples of pairs $(c_2, c_1^2)$ occuring with $\Omega^1_X$ big

- $D_{2,k}$ are examples coming from Debarre const. $(b_1 \neq 0)$
- $(2:1)$ is the example of double cover of $\mathbb{P}^2$ branched along a curve of degree 16 with 24 $a_7$ sing. $(b_1 = 0)$
- The $\times$ 6, $\times$ 7 are the $(c_2, c_1^2)$ for hyp. of degree 6, 7 while 8 for degree 8 can have $\Omega^1_X$ big.

Fim